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A Class of Operators by Means of Three-Diagonal Matrices

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We give a method of linear approximation by means of three-diagonal matrices. By this method we modify the Kantorovich operators and obtain a new class of operators which overcomes a difficulty in extending a Berens–Lorentz result to the Kantorvich operators for second order of smoothness. The direct and inverse theorems for these operators in L_p are also presented by the Ditzian–Totik modulus of smoothness.

1. INTRODUCTION

For $f \in C[0, 1]$ the nth Bernstein polynomial is defined by

$$B_n(f, x) = \sum_{k=0}^{n} P_{n,k}(x) f\left(\frac{k}{n}\right) \equiv \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right). \quad (1.1)$$

It was shown by H. Berens and G. G. Lorentz [2] in 1972 that if $0 < \alpha < 2$ then $\omega_2(f, t) = O(t^\alpha)$ if and only if $|B_n(f, x) - f(x)| \leq M(x(1-x)/n)^{\alpha/2}$. Here $\omega_2(f, t)$ is the classical modulus of smoothness defined by

$$\omega_2(f, t) = \sup_{0<h\leq t} \|\Delta_h^2 f(x)\|_{C[0,1]},$$

$$\begin{cases} \Delta_h^2 f(x) = f(x+h) - 2f(x) + f(x-h), & \text{if } x \pm h \in [0, 1]; \\ \Delta_h^2 f(x) = 0, & \text{otherwise.} \end{cases}$$

Compared with the above result we showed in [9] for the Kantorovich

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operators

$$K_n(f,x) = \sum_{k=0}^{n} P_{n,k}(x)(n+1)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\,dt \tag{1.2}$$

that if $0 < \alpha < 1$ then $\omega_2(f,t) = O(t^\alpha)$ if and only if $|K_n(f,x) - f(x)| \le M(x(1-x)/n + 1/n^2)^{\alpha/2}$. Here the term $x(1-x)/n + 1/n^2$ cannot be replaced by $x(1-x)/n$. We also showed [9] that if $1 < \alpha < 2$ then there exist no functions $\{\varphi_{n,\alpha}(x)\}_{n\in N}$ such that

$$\omega_2(f,t) = O(t^\alpha) \Leftrightarrow |K_n(f,x) - f(x)| \le M\varphi_{n,\alpha}(x). \tag{1.3}$$

Thus, we cannot characterize the second orders of Lipschitz functions by the rate of convergence for the Kantorovich operators. To overcome this difficulty, S. M. Mazhar and V. Totik [6] introduced a method to modify integral operators that do not reproduce linear functions. This modification is valid for many classes of Bernstein type integral operators and makes it possible to characterize the second orders of smoothness by these operators. However, these modified Bernstein type operators have the drawback that they are not suitable for $L_p(1 \le p < \infty)$-approximation.

In this paper we introduce a new method of linear approximation by means of matrices.

Let $\{e_i(x)\}_{i=0}^n$ be a basis of a subspace P of $C[a,b]$, and $\{L_i\}_{i=0}^n$ be some functionals on $C[a,b]$. We denote $e(x) = (e_0(x), \ldots, e_n(x))$, $L = (L_0, \ldots, L_n)^T$. For any $(n+1)\times(n+1)$ matrix A we can construct a linear operator from $C[a,b]$ to P as

$$T(f,x) = e(x)AL(f). \tag{1.4}$$

By choosing suitable bases and functionals we can obtain different kinds of operators for different kinds of tasks. General properties of this method and some other applications will be discussed elsewhere. Here we only discuss some applications in Bernstein type operators.

We choose $e(x) = (P_{n,0}(x), \ldots, P_{n,n}(x)) \equiv P_n(x)$, $L_{n,i}f = (n+1)\int_{i/(n+1)}^{(i+1)/(n+1)} f(t)\,dt$. Then

$$T_n(f,x) = P_n(x)A_nL_n(f) \tag{1.5}$$

is the Bernstein type operator we want to discuss. We will show for a suitable series of matrices $\{A_n\}_{n\in N}$ that if $0 < \alpha < 2$ then

$$\omega_2(f,t) = O(t^\alpha) \Leftrightarrow |T_n(f,x) - f(x)| \le M\left(\frac{x(1-x)}{n} + \frac{1}{n^2}\right)^{\alpha/2}. \tag{1.6}$$

That is, the difficulty in extending the Berens–Lorentz result to the Kantorovich operators for the second order of smoothness can be overcome by our method.

These operators have the advantage that they are approximation processes in $L_p[0,1]$ $(1 < p < \infty)$. We give the direct and inverse theorems for these operators in $L_p[0,1]$.

2. Construction of the Operators

Let $A_n = (a_{i,j})_{i,j=0}^{n}$ be an $(n+1) \times (n+1)$ matrix which will be determined later. To simplify our discussion we make a restriction that $a_{i,j} = 0$ for $|i-j| \geq 2$. In order that the operators $\{T_n\}_{n \in N}$ defined by (1.5) satisfy (1.6) we need the assumptions

$$T_n(1,x) = 1; \tag{2.1}$$

$$T_n(t,x) = x; \tag{2.2}$$

$$T_n\big((t-x)^2, x\big) \leq M\left(\frac{x(1-x)}{n} + \frac{1}{n^2}\right). \tag{2.3}$$

Therefore, we can now choose $\{A_n\}_{n \in N}$ as follows.

THEOREM 2.1. *Let $n \in N$, A_n, and T_n be defined as above. Then the assumptions* (2.1) *and* (2.2) *are satisfied if and only if*

$$\sum_{j=0}^{n} a_{i,j} = 1, \qquad \text{for } i = 1, \ldots, n-1 \tag{2.4}$$

$$\begin{aligned} a_{i,i+1} &= a_{i,i-1} + \frac{2i-n}{2n}, \qquad \text{for } i = 1, 2, \ldots, n-1; \\ a_{0,0} &= \tfrac{3}{2} = a_{n,n}; \\ a_{0,1} &= -\frac{1}{2} = a_{n,n-1}. \end{aligned} \tag{2.5}$$

Proof. Note that $L_{n,i}(1) = 1$, $L_{n,i}(t) = (2i+1)/2(n+1)$ for $0 \leq i \leq n$, and $a_{i,j} = 0$ for $|i-j| \geq 2$. We have

$$T_n(1,x) = P_n(x)\left(\sum_{|j-i| \leq 1} a_{i,j}\right)^T$$

and

$$T_n(t,x) = P_n(x)\Bigg(a_{0,0}\frac{1}{2(n+1)} + a_{0,1}\frac{3}{2(n+1)}, \quad \dots,$$
$$\left(a_{i,i-1}\frac{2i-1}{2(n+1)} + a_{i,i}\frac{2i+1}{2(n+1)} + a_{i,i+1}\frac{2i+3}{2(n+1)}\right)_{i=1}^{n-1},$$
$$\dots, \quad a_{n,n-1}\frac{2n-1}{2(n+1)} + a_{n,n}\frac{2i+1}{2(n+1)}\Bigg)^T.$$

We observe from [3] or [4, Chap. 9] that $\{P_{n,0},\dots,P_{n,n}\}$ is another basis of the liner space span$\{1, x, \dots, x^n\}$, which implies that they are linearly independent. That is, for any two $(n+1)$-vectors A and B, $P_n A^T = P_n B^T$ is equivalent to $A = B$. Therefore, from the expressions $1 = P_n(x)(1,\dots,1)^T$ and $x = P_n(x)(0/n,\dots,n/n)^T$ we know that Theorem 2.1 holds. The proof is complete.

We denote $a_{i,i} = a_i$, $a_{i,i+1} = b_i$, $a_{i,i-1} = c_i$, and set $c_0 = b_n = 0$. Then we can estimate (2.3) as follows

THEOREM 2.2. *Let A_n, T_n be defined as above and satisfy* (2.4) *and* (2.5). *Then we have*

$$T_n(t^2,x) = \left(1+\frac{2}{n}\right)\frac{n^2}{(n+1)^2}\left(x^2+\frac{x(1-x)}{n}\right) + \sum_{i=1}^{n-1}\frac{2b_i}{(n+1)^2}P_{n,i}(x) + \frac{1}{3(n+1)^2} - \frac{1}{(n+1)^2}P_{n,0}(x) \tag{2.6}$$

and

$$T_n\big((t-x)^2,x\big) \le \frac{x(1-x)}{n+1} + (n+1)^{-2}\left(\frac{4}{3} + 2\sup_{1\le i\le n-1}|b_i|\right). \tag{2.7}$$

Proof. We observe that for $0 \le i \le n$

$$L_{n,i}(t^2) = \frac{3i^2+3i+1}{3(n+1)^2}.$$

Then we have

$$
\begin{aligned}
T_n(t^2, x) &= \sum_{i=0}^{n} P_{n,i}(x) a_{i,j} L_{n,j}(t^2) \\
&= \sum_{i=0}^{n} P_{n,i}(x)\{c_i L_{n,i-1}(t^2) + a_i L_{n,i}(t^2) + b_i L_{n,i+1}(t^2)\} \\
&= P_{n,0}(x)\big(a_0 L_{n,0}(t^2) + b_0 L_{n,1}(t^2)\big) \\
&\quad + P_{n,n}(x)\big(a_n L_{n,n}(t^2) + c_n L_{n,n-1}(t^2)\big) \\
&\quad + \sum_{i=1}^{n-1} P_{n,i}(x)\left\{c_i \frac{3(i-1)^2 + 3(i-1) + 1}{3(n+1)^2}\right. \\
&\qquad \left. + a_i \frac{3i^2 + 3i + 1}{3(n+1)^2} + b_i \frac{3(i+1)^2 + 3(i+1) + 1}{3(n+1)^2}\right\} \\
&= \sum_{i=0}^{n} P_{n,i}(x) d_{n,i}.
\end{aligned}
$$

Here

$$
\begin{aligned}
d_{n,0} &= -\frac{2}{3(n+1)^2}; \\
d_{n,n} &= \frac{3n^2 + 6n + 1}{3(n+1)^2}; \\
d_{n,i} &= \frac{3i^2(1 + 2/n) + 1 + 6b_i}{3(n+1)^2}.
\end{aligned}
$$

Thus we obtain (2.6) and

$$
\begin{aligned}
T_n\big((t-x)^2, x\big) &= T_n(t^2, x) - x^2 \\
&= \frac{x(1-x)}{n+1} + (n+1)^{-2}\left(x(1-x) - x^2 + \frac{1 - 3(1-x)^n}{3}\right. \\
&\qquad \left. + \sum_{i=1}^{n-1} 2 b_i P_{n,i}(x)\right) \\
&\le \frac{x(1-x)}{n+1} + (n+1)^{-2}\left(\frac{4}{3} + 2 \sup_{1 \le i \le n-1} |b_i|\right).
\end{aligned}
$$

The proof of Theorem 2.2 is complete.

Now we determine the matrix A_n by

$$0 \le a_i, b_i, c_i \le 1, \qquad \text{for } 1 \le i \le n-1. \tag{2.8}$$

This is possible since we have

$$c_i \ge 0$$
$$b_i = c_i + \frac{2i-n}{2n} \ge 0$$
$$a_i = 1 - 2c_i - \frac{2i-n}{2n} \ge 0$$

if we choose

$$\frac{3}{4} - \frac{i}{2n} \ge c_i \ge \max\left\{0, \frac{n-2i}{2n}\right\}. \tag{2.9}$$

The restriction (2.8) on the matrix A_n makes the operator T_n defined by (1.5) "almost positive."

Remark. Other choices are possible to make our following main results valid.

3. Main Results

The matrix A_n and the corresponding operators $\{T_n\}_{n \in N}$ are determined in Section 2. We can now state our main results. In what follows of this paper we assume that the matrix satisfies (2.4), (2.5), and (2.8).

The Berens–Lorentz type Theorem for our operators can be given in the following

Theorem 1. *Let $\{T_n\}_{n \in N}$ be defined as above, $f \in C[0,1]$, $0 < \alpha < 2$. Then we have*

$$|T_n(f,x) - f(x)| \le M\left(\frac{x(1-x)}{n} + \frac{1}{n^2}\right)^{\alpha/2} \tag{3.1}$$

if and only if

$$\omega_2(f,t) = O(t^\alpha). \tag{3.2}$$

Our operators are also valid for L_p-approximations. In fact, we have the following direct and converse result.

THEOREM 2. *Let* $\{T_n\}_{n \in N}$ *be defined as above,* $f \in L_p$ *if* $1 \le p < \infty$, *or* $f \in C[0,1]$ *if* $p = \infty$, $0 < \alpha < 1$, $\varphi(x) = \sqrt{x(1-x)}$. *Then we have*

$$\|T_n f - f\|_p = O(n^{-\alpha}) \tag{3.3}$$

if and only if

$$\omega_\varphi^2(f,t)_p = O(t^{2\alpha}), \tag{3.4}$$

where $\omega_\varphi^2(f,t)_p = \sup_{0<h\le t}\|\Delta_{h\varphi(x)}^2 f(x)\|_p$ *is the Ditzian–Totik modulus of smoothness* [4].

Remark. It must be interesting to extend our theorems to higher orders of smoothness if we choose other kinds of matrices and corresponding operators. However, we conjecture that this will not be true.

To prove our theorems we need some lemmas first.

4. LEMMAS

Explicitly, we have

$$T_n(f,x) = \sum_{i=0}^{n} P_{n,i}(x)\bigl(c_i L_{n,i-1}(f) + a_i L_{n,i}(f) + b_i L_{n,i+1}(f)\bigr). \tag{4.1}$$

The main tools for the proof of the inverse parts of Theorems 1 and 2 are some Bernstein type inequalities.

LEMMA 4.1. *Let* $T_n(f,x)$ *be defined as above,* $n \in N$, $1 \le p \le \infty$, $f \in L_p[0,1]$. *Then we have*

$$\|T_n f\|_p \le 9\|f\|_p, \tag{4.2}$$

$$\|T_n''(f)\|_p \le M_1 n^2 \|f\|_p, \tag{4.3}$$

$$\|\varphi^2 T_n''(f)\|_p \le M_1 n \|f\|_p, \tag{4.4}$$

where M_1 *is a constant independent of* f *and* n.

Proof. Note that $0 \le a_i, b_i, c_i \le 1$, for $1 \le i \le n-1$, and $\int_0^1 P_{n,i}(x)\,dx = 1/(n+1)$. For $p = 1$, we have from (4.1)

$$\begin{aligned}\|T_n f\|_1 &\le \sum_{i=0}^{n} \int_0^1 P_{n,i}(x)\,dx\big(|c_i|\,|L_{n,i-1}(f)| \\ &\qquad + |a_i|\,|L_{n,i}(f)| + |b_i|\,|L_{n,i+1}(f)|\big) \\ &\le \frac{1}{n+1}\sum_{i=0}^{n} 3|L_{n,i}(f)| \\ &\le \frac{3}{n+1}\sum_{i=0}^{n}(n+1)\int_{i/(n+1)}^{(i+1)/(n+1)}|f(t)|\,dt \\ &= 3\|f\|_1.\end{aligned}$$

If $1 < p < \infty$, then by Hölder's inequality we have

$$\begin{aligned}\|T_n f\|_p^p &\le \sum_{i=0}^{n} \int_0^1 P_{n,i}(x)\,dx\big(|c_i|\,|L_{n,i-1}(f)| \\ &\qquad + |a_i|\,|L_{n,i}(f)| + |b_i|\,|L_{n,i+1}(f)|\big)^p \\ &\le \frac{1}{n+1}\sum_{i=0}^{n} 9^p|L_{n,i}(f)|^p \\ &\le \frac{9^p}{n+1}\sum_{i=0}^{n}(n+1)^p\int_{i/(n+1)}^{(i+1)/(n+1)}|f(t)|^p\,dt\left(\frac{1}{n+1}\right)^{p-1} \\ &\le 9^p\|f\|_p^p.\end{aligned}$$

The case $p = \infty$ can be proved in the same way. In fact, for $x \in (0,1)$ we have

$$\begin{aligned}|T_n(f,x)| &\le \sum_{i=0}^{n} P_{n,i}(x)\big(|c_i|\,|L_{n,i-1}(f)| + |a_i|\,|L_{n,i}(f)| \\ &\qquad + |b_i|\,|L_{n,i+1}(f)|\big) \\ &\le 3\|f\|_\infty.\end{aligned}$$

Therefore, for $1 \le p \le \infty$ we have

$$\|T_n f\| \le 9\|f\|_p,$$

which completes the proof of (4.2).

We denote $d_{n,i}(f) = c_i L_{n,i-1}(f) + a_i L_{n,i}(f) + b_i L_{n,i+1}(f)$. Then we have

$$T_n''(f,x) = n(n-1)\sum_{i=0}^{n-2} P_{n-2,i}(x)\big(d_{n,i+2}(f) - 2d_{n,i+1}(f) + d_{n,i}(f)\big). \tag{4.5}$$

Hence

$$|T_n''(f,x)| \le 9n^2 \sum_{i=0}^{n-2} P_{n-2,i}(x) \sum_{\substack{i-1\le j\le i+3\\ 0\le j\le n}} L_{n,j}(|f|).$$

Therefore, we can estimate (4.3) as follows.

For $p=\infty$, we have

$$\begin{aligned}|T_n''(f,x)| &\le 9n^2 \sum_{i=0}^{n-2} P_{n-2,i}(x) 5\|f\|_\infty\\ &\le 45n^2\|f\|_\infty.\end{aligned}$$

For $1 \le p < \infty$, we have by Hölder's inequality

$$\begin{aligned}\|T_n''(f,x)\|_p^p &\le (9n^2)^p \sum_{i=0}^{n-2} \int_0^1 P_{n-2,i}(x)\, dx 5^p\\ &\qquad \sum_{\substack{i-1\le j\le i+3\\ 0\le j\le n}} (n+1)^p (n+1)^{1-p} \int_{j/(n+1)}^{(j+1)/(n+1)} |f(t)|^p\, dt\\ &\le (45n^2)^p 15\|f\|_p^p.\end{aligned}$$

Hence

$$\|T_n''(f,x)\|_p \le 675n^2\|f\|_p.$$

By the standard method in [4], the proof of (4.4) is easy and we omit it here. Our proof of Lemma 4.1 is complete.

LEMMA 4.2. *Let $n \in N$, $T_n(f,x)$ be defined as above, $1 \le p \le \infty$, $f \in L_p[0,1]$, $f, f' \in A.C.loc$. Then we have*

$$\|T_n''(f,x)\|_p \le M_2\|f''\|_p, \tag{4.6}$$

$$\|\varphi^2 T_n''(f,x)\|_p \le M_2\|\varphi^2 f''\|_p, \tag{4.7}$$

where M_2 is a constant independent of f and n.

Proof. If we denote

$$\Delta L_{n,k}(f) = L_{n,k+1}(f) - L_{n,k}(f),$$
$$\Delta^2 L_{n,k}(f) = L_{n,k+2}(f) - 2L_{n,k+1}(f) + L_{n,k}(f),$$

we have by (4.5)

$$\begin{aligned}
T_n''(f,x) &= n(n-1)\sum_{i=0}^{n-2} P_{n-2,i}(x) \\
&\quad \times \{c_{i+2}L_{n,i+1}(f) \\
&\qquad + (1-b_{i+2}-c_{i+2})L_{n,i+2}(f) + b_{i+2}L_{n,i+3}(f) \\
&\qquad - 2c_{i+1}L_{n,i}(f) - 2(1-c_{i+1}-b_{i+1})L_{n,i+1}(f) \\
&\qquad - 2b_{i+1}L_{n,i+2}(f) + c_iL_{n,i-1}(f) \\
&\qquad + (1-c_i-b_i)L_{n,i}(f) + b_iL_{n,i+1}(f)\} \\
&= n(n-1)\sum_{i=0}^{n-2} P_{n-2,i}(x)\Bigg\{c_{i+2}\Delta^2 L_{n,i+1}(f) \\
&\quad + \left(1 - \frac{2(i+2)-n}{2n}\right)L_{n,i+2}(f) + \frac{2(i+2)-n}{2n}L_{n,i+3}(f) \\
&\quad - 2c_{i+1}\Delta^2 L_{n,i}(f) - 2\left(1 - \frac{2(i+1)-n}{2n}\right)L_{n,i+1}(f) \\
&\quad - 2\frac{2(i+1)-n}{2n}L_{n,i+2}(f) + c_i\Delta^2 L_{n,i-1}(f) \\
&\quad + \left(1 - \frac{2i-n}{2n}\right)L_{n,i}(f) + \frac{2i-n}{2n}L_{n,i+1}(f)\Bigg\} \\
&= n(n-1)\sum_{i=0}^{n-2} P_{n-2,i}(x) \\
&\quad \times \Bigg\{\Delta^2 L_{n,i}(f) + c_{i+2}\Delta^2 L_{n,i+1}(f) \\
&\qquad - 2c_{i+1}\Delta^2 L_{n,i}(f) + c_i\Delta^2 L_{n,i-1}(f) \\
&\qquad + \left(\frac{i}{n} - \frac{1}{2}\right)\left(\Delta^2 L_{n,i+1}(f) - \Delta^2 L_{n,i}(f)\right) \\
&\qquad + \frac{2}{n}\Delta^2 L_{n,i+1}(f)\Bigg\}.
\end{aligned} \tag{4.8}$$

From this formula we prove (4.6) and (4.7). By the Riesz–Thorin Theorem it is sufficient to prove the estimates in the cases $p = 1$ and $p = \infty$.

For $p = \infty$, we observe that for $0 \le i \le n - 2$

$$|\Delta^2 L_{n,i}(f)| = \left|(n+1)\int_{i/(n+1)}^{(i+1)/(n+1)} \iint_0^{1/(n+1)} f''(t+u+v)\,du\,dv\,dt\right| \le \frac{\|f''\|_\infty}{(n+1)^2}$$

and for $n \ge 7$

$$\begin{aligned}
|\Delta^2 L_{n,i}(f)| &\le (n+1)\int_{i/(n+1)}^{(i+1)/(n+1)} \iint_0^{1/(n+1)} \frac{1}{\varphi^2(t+u+v)}\,du\,dv\,dt\,\|\varphi^2 f''\|_\infty \\
&\le (n+1)\int_{i/(n+1)}^{(i+1)/(n+1)} \frac{6(n+1)^{-2}}{\varphi^2(t+1/(n+1))}\,dt\,\|\varphi^2 f''\|_\infty \\
&\le 6(n+1)^{-2}\frac{n+1}{i+1}\,\frac{n+1}{n-i-1}\|\varphi^2 f''\|_\infty \\
&\le 6\frac{1}{(i+1)(n-i-1)}\|\varphi^2 f''\|_\infty .
\end{aligned}$$

Here we made use of an inequality of M. Becker in [1]

$$\iint_{-h/2}^{h/2} \frac{1}{\varphi^2(x+u+v)}\,du\,dv \le \frac{6h^2}{\max\{\varphi^2(x\pm h), \varphi^2(x)\}}, \qquad h \le \frac{1}{8}. \tag{4.9}$$

Thus, we have for $x \in (0,1)$

$$\begin{aligned}
|T_n''(f,x)| &\le n(n-1)\sum_{i=0}^{n-2} P_{n-2,i}(x)(n+1)^{-2} 9\|f''\|_\infty \\
&\le 9\|f''\|_\infty .
\end{aligned}$$

Hence

$$\|T_n''(f,x)\|_\infty \le 9\|f''\|_\infty .$$

For $x \in (0,1)$ we also have

$$\begin{aligned}
&|\varphi^2(x) T_n''(f,x)| \\
&\quad \le n(n-1) \sum_{i=0}^{n-2} x(1-x) P_{n-2,i}(x) 6 \|\varphi^2 f''\|_\infty \\
&\qquad \times \left\{ \frac{4}{(i+1)(n-i-1)} + \frac{|c_i|}{i(n-i)} \right. \\
&\qquad\qquad \left. + \frac{1}{(i+2)(n-i-2)} \left| c_{i+2} + \frac{i}{n} - \frac{1}{2} + \frac{2}{n} \right| \right\} \\
&\quad \le 6 \sum_{i=0}^{n-2} P_{n,i+1}(x)(i+1)(n-i-1) \|\varphi^2 f''\|_\infty \frac{36}{(i+1)(n-i-1)} \\
&\quad \le M_2 \|\varphi^2 f''\|_\infty .
\end{aligned}$$

Here for simplicity we have denoted $|c_0|/(0 \cdot n)$, $|c_n + 1/2|/(n \cdot 0)$ as 0. Thus we have proved (4.6) and (4.7) for $p = \infty$.

For $p = 1$, we have

$$\begin{aligned}
\|T_n''(f,x)\|_1 &\le n(n-1) \sum_{i=0}^{n-2} \frac{1}{n-1} \\
&\quad \times \left\{ 4(n+1) \int_{i/(n+1)}^{(i+1)/(n+1)} \int \int_0^{1/(n+1)} |f''(t+u+v)| \, du \, dv \, dt \right. \\
&\qquad + \left| c_{i+2} + \frac{i}{n} - \frac{1}{2} + \frac{2}{n} \right| (n+1) \int_{(i+1)/(n+1)}^{(i+2)/(n+1)} \\
&\qquad \times \int \int_0^{1/(n+1)} |f''(t+u+v)| \, du \, dv \, dt \\
&\qquad \left. + |c_i| (n+1) \int_{(i-1)/(n+1)}^{i/(n+1)} \int \int_0^{1/(n+1)} |f''(t+u+v)| \, du \, dv \, dt \right\} \\
&\le 8n(n+1) \int \int_0^{1/(n+1)} 3 \int_0^{(n-1)/(n+1)} |f''(t+u+v)| \, dt \, du \, dv \\
&\le 24 \|f''\|_1 .
\end{aligned}$$

We also have

$$\|\varphi^2 T_n''(f,x)\|_1 \le \sum_{i=0}^{n-2} \frac{(i+1)(n-i-1)}{n+1}$$

$$\left\{4(n+1)\int_{i/(n+1)}^{(i+1)/(n+1)}\int\int_0^{1/(n+1)}|f''(t+u+v)|\,du\,dv\,dt\right.$$

$$+\left|c_{i+2}+\frac{i}{n}-\frac{1}{2}+\frac{2}{n}\right|(n+1)\int_{(i+1)/(n+1)}^{(i+2)/(n+1)}$$

$$\times\int\int_0^{1/(n+1)}|f''(t+u+v)|\,du\,dv\,dt$$

$$\left.+|c_i|(n+1)\int_{(i-1)/(n+1)}^{i/(n+1)}\int\int_0^{1/(n+1)}|f''(t+u+v)|\,du\,dv\,dt\right\}$$

$$:= I + J + K.$$

Let us estimate the first term. For $n \ge 3$ we have by (4.9)

$$I \le 4\sum_{i=0}^{n-2}(i+1)(n-i-1)\int_{i/(n+1)}^{(i+1)/(n+1)}\int\int_0^{1/(n+1)}$$

$$\times\frac{1}{(i/(n+1)+u+v)(1-(i+1)/(n+1)-u-v)}$$

$$|\varphi^2 f''(t+u+v)|\,du\,dv\,dt$$

$$\le 4\sum_{i=0}^{n-2}(i+1)(n-i-1)\int_{i/(n+1)}^{(i+3)/(n+1)}|\varphi^2 f''(\omega)|\,d\omega$$

$$\int\int_0^{1/(n+1)}\left\{\frac{2}{\varphi^2(i/(n+1)+u+v)}\right.$$

$$\left.+\frac{2}{\varphi^2((i+1)/(n+1)+u+v)}\right\}du\,dv$$

$$\le 48\sum_{i=0}^{n-2}\frac{(i+1)(n-i-1)}{(n+1)^2}\int_{i/(n+1)}^{(i+3)/(n+1)}$$

$$\times|\varphi^2 f''(\omega)|\,d\omega\,\frac{2}{\varphi^2((i+1)/(n+1))}$$

$$\le 96\sum_{i=0}^{n-2}\int_{i/(n+1)}^{(i+3)/(n+1)}|\varphi^2 f''(\omega)|\,d\omega$$

$$\le 288\|\varphi^2 f''\|_1.$$

The estimates of J and K can be obtained in the same way and we get

$$\|\varphi^2 T_n''\|_1 \le M_2 \|\varphi^2 f''\|_1.$$

Thus, (4.7) is also valid for $p = 1$. Our proof of Lemma 4.2 is complete.

With the above lemmas we can now prove our main results.

5. Proof of Theorem 1

To prove Theorem 1 we need the Peetre K-functional given by

$$K_2(f, t) = \inf_{g \in C^2[0,1]} \{\|f - g\|_\infty + t\|g''\|_\infty\}. \tag{5.1}$$

This K-functional is equivalent to the classical modulus of smoothness in the sense that

$$M_0^{-1}\omega_2(f, t) \le K_2(f, t^2) \le M_0 \omega_2(f, t) \tag{5.2}$$

with a positive constant M_0 independent of $f \in C[0, 1]$ and $t > 0$ (see [4]).

By this K-functional our proof of the direct part is simple.

Proof of Theorem 1. Sufficiency. Suppose that $\omega_2(f, t) \le Mt^\alpha$. By (5.2) we have $K_2(f, t) \le M_0 M t^{\alpha/2}$.

Let $x \in (0, 1)$, $n \in N$, $g \in C^2[0, 1]$. We have

$$\begin{aligned} |T_n(g, x) - g(x)| &= \left| T_n\left(g'(x)(t - x) + \int_x^t (t - u) g''(u)\, du, x \right) \right| \\ &= \left| T_n\left(\int_x^t (t - u) g''(u)\, du, x \right) \right|. \end{aligned}$$

The operator T_n is "almost positive" in the sense that $a_{i,j} \ge 0$ except $a_{0,1} = b_0 = a_{n,n-1} = c_n = -1/2$. But it is not positive. So we define a new class of positive operators on $C[0, 1]$ as

$$T_n^-(f, x) = P_{n,0}(x)|b_0| L_{n,1}(f) + P_{n,n}(x)|c_n| L_{n,n-1}(f).$$

Then we know that $T_n + T_n^-$ is positive. Therefore, by (2.7) we have

$$\left|T_n\left(\int_x^t (t-u)g''(u)\,du, x\right)\right|$$
$$\le (T_n + T_n^-)\left(\left|\int_x^t (t-u)g''(u)\,du\right|, x\right) + T_n^-\left(\left|\int_x^t (t-u)g''(u)\,du\right|, x\right)$$
$$\le \left\{(T_n + T_n^-)\left((t-x)^2, x\right) + T_n^-\left((t-x)^2, x\right)\right\}\|g''\|_\infty$$
$$= \left\{T_n\left((t-x)^2, x\right) + 2T_n^-\left((t-x)^2, x\right)\right\}\|g''\|_\infty$$
$$\le \left\{\frac{x(1-x)}{n} + \frac{4}{n^2} + P_{n,0}(x)(n+1)\right.$$
$$\left.\times \int_{1/(n+1)}^{2/(n+1)} (t-x)^2\,dt + P_{n,n}(x)(n+1)\int_{(n-1)/(n+1)}^{n/(n+1)} (t-x)^2\,dx\right\}\|g''\|_\infty$$
$$\le \left\{\frac{x(1-x)}{n} + \frac{4}{n^2} + 3\left(\frac{4}{n^2} + x^2\right)P_{n,0}(x)\right.$$
$$\left. +3\left(\frac{4}{n^2} + (1-x)^2\right)P_{n,n}(x)\right\}\|g''\|_\infty$$
$$\le \left\{\frac{x(1-x)}{n} + \frac{28}{n^2} + \frac{6}{n^2}P_{n+2,2}(x) + \frac{6}{n^2}P_{n+2,n}(x)\right\}\|g''\|_\infty$$
$$\le 40\left(\frac{x(1-x)}{n} + \frac{1}{n^2}\right)\|g''\|_\infty.$$

Thus, by (4.2) we obtain

$$|T_n(f,x) - f(x)|$$
$$\le \inf_{g\in C^2[0,1]} \{\|T_n(f-g)\|_\infty + \|f-g\|_\infty + |T_n(g,x) - g(x)|\}$$
$$\le 40 \inf_{g\in C^2[0,1]} \left\{\|f-g\|_\infty + \left(\frac{x(1-x)}{n} + \frac{1}{n^2}\right)\|g''\|_\infty\right\}$$
$$= 40K_2\left(f, \frac{x(1-x)}{n} + \frac{1}{n^2}\right)$$
$$\le 40M_0M\left(\frac{x(1-x)}{n} + \frac{1}{n^2}\right)^{\alpha/2}.$$

The proof of the sufficiency is complete.

Necessity. Suppose that (3.1) holds. We want to show that for some positive constant M' independent of t, $\delta \in (0, 1/8)$

$$\omega_2(f, t) \le M'\{\delta^\alpha + (t/\delta)^2 \omega_2(f, \delta)\} \tag{5.3}$$

which implies (3.2) by the Berens–Lorentz Lemma [2, 4].

Let $0 < h \le t < 1/8$, $n \in N$, $x \in (0, 1)$, $x \pm h \in (0, 1)$. We denote

$$d(n, x, h) = \max\left\{\frac{1}{n}, \frac{\varphi(x+h)}{\sqrt{n}}, \frac{\varphi(x-h)}{\sqrt{n}}, \frac{\varphi(x)}{\sqrt{n}}\right\}.$$

Then we have

$$\begin{aligned} |\Delta_h^2 f(x)| &\le |\Delta_h^2(f - T_n f)(x)| + |\Delta_h^2(T_n f)(x)| \\ &\le 8M(d(n, x, h))^\alpha + \int\int_{-h/2}^{h/2} |T_n''(f - f_d, x + u + v)|\, du\, dv \\ &\quad + \int\int_{-h/2}^{h/2} |T_n''(f_d, x + u + v)|\, du\, dv. \end{aligned} \tag{5.4}$$

Here $f_d \in C^2[0, 1]$ is taken, for any $d > 0$, such that

$$\begin{aligned} \|f - f_d\|_\infty &\le 2K_2(f, d^2) \le 2M_0 \omega_2(f, d), \\ \|f_d''\|_\infty &\le 2d^{-2} K_2(f, d^2) \le 2M_0 d^{-2} \omega_2(f, d). \end{aligned} \tag{5.5}$$

By (5.2) this is possible.

Now we want to estimate the terms in (5.4). By (4.3) and (4.4) we have

$$\begin{aligned} &\int\int_{-h/2}^{h/2} |T_n''(f - f_d, x + u + v)|\, du\, dv \\ &\le \min\left\{M_1 n^2 h^2 \|f - f_d\|_\infty, M_1 n \|f - f_d\|_\infty \int\int_{-h/2}^{h/2} \frac{1}{\varphi^2(x + u + v)}\, du\, dv\right\} \\ &\le M_1 \|f - f_d\|_\infty \min\left\{n^2 h^2, n \frac{6h^2}{\max\{\varphi^2(x \pm h), \varphi^2(x)\}}\right\} \\ &\le 6M_1 \|f - f_d\|_\infty h^2 (d(n, x, h))^{-2}. \end{aligned}$$

Here we used the estimate (4.9).

By (4.6) we also have

$$\int\int_{-h/2}^{h/2} |T_n''(f_d, x+u+v)|\,du\,dv \le M_2 \|f_d''\|_\infty h^2.$$

Therefore, combining the above estimates we get

$$|\Delta_h^2 f(x)| \le 8M(d(n,x,h))^\alpha + 12 M_0 M_1 h^2 (d(n,x,h))^{-2} \omega_2(f,d) + 2M_0 M_2 h^2 d^{-2} \omega_2(f,d).$$

Let $d = d(n, x, h)$. We obtain

$$|\Delta_h^2 f(x)| \le 8M(d(n,x,h))^\alpha + 12 M_0 (M_1 + M_2) h^2 (d(n,x,h))^{-2} \omega_2(f, d(n,x,h)).$$

We observe that the sequence $d(n, x, h)$ decreases to zero and satisfies $d(n,x,h) \le d(n-1,x,h) \le 2d(n,x,h)$. For any $\delta \in (0, 1/8)$, we can choose an integer $n \in N$ such that $d(n,x,t) \le \delta < 2d(n,x,t)$.

Then we have

$$|\Delta_h^2 f(x)| \le 8M\delta^\alpha + 48 M_0 (M_1 + M_2)(h/\delta)^2 \omega_2(f,\delta).$$

Here the constants are independent of x and h. Hence

$$\omega_2(f,t) \le 8M\delta^\alpha + 48 M_0 (M_1 + M_2)(t/\delta)^2 \omega_2(f,\delta).$$

Therefore, (5.3) holds. Our proof of Theorem 1 is complete.

Remark. We do not know if Theorem 1 holds for $\alpha = 2$. Also we want to know if the term $x(1-x)/n + 1/n^2$ can be replaced by $x(1-x)/n$ for some special choice of the matrices.

6. Proof of Theorem 2

To prove our Theorem 2 we use the Ditzian–Totik K-functionals. These K-functionals and the corresponding modulus of smoothness have many applications concerning positive linear operators, best approximation of algebraic polynomials, and embedding problems. In the cases of $1 \le p \le \infty$ the K-functional is defined by

$$K_{\varphi,2}(f,t^2)_p = \inf_{g' \in A.C._{loc}} \{\|f-g\|_p + t^2(\|g\|_p + \|\varphi^2 g''\|_p)\}. \quad (6.1)$$

This K-functional is equivalent to the Ditzian–Totik modulus of smoothness in the sense that

$$M_p^{-1}\omega_\varphi^2(f,t)_p \le K_{\varphi,2}(f,t^2)_p \le M_p\omega_\varphi^2(f,t)_p \tag{6.2}$$

for some constant M_p depending only on p. By these tools we can prove our Theorem 2 in the case $1 \le p < \infty$ by a standard method.

Proof of Theorem 2 *for* $1 \le p < \infty$. By Lemma 4.1, Lemma 4.2, (6.2), and a result of A. Grundmann [5] (see also [4]) it is sufficient to show that if $g' \in A.C._{loc}$, then

$$\|T_n g - g\|_p \le \frac{C_p}{n}\left(\|g\|_p + \|\varphi^2 g''\|_p\right), \tag{6.3}$$

where C_p is a constant independent of g and n.

We use a result of Totik in [7] that if $g \in C^2[0,1]$, then

$$\|B_n g - g\|_p \le \frac{C_p}{n}\left(\|g\|_p + \|\varphi^2 g''\|_p\right).$$

Since $C^2[0,1]$ is dense in the weighted Sobolev space $D_p = \{g \in L_p[0,1]: g' \in A.C._{loc}, \|g\|_{D_p} = \|g\|_p + \|\varphi^2 g''\|_p < \infty\}$, it is enough to prove that for $g \in C^2[0,1]$

$$\|B_n g - T_n g\|_p \le \frac{C_p}{n}\left(\|g\|_p + \|\varphi^2 g''\|_p\right). \tag{6.4}$$

Note that

$$\begin{aligned} &T_n(g,x) - B_n(g,x) \\ &\quad = \sum_{i=0}^{n} P_{n,i}(x)\Bigg(c_i(n+1)\int_{(i-1)/(n+1)}^{i/(n+1)}\left[g(t) - g\left(\frac{i}{n}\right)\right]dt \\ &\qquad + a_i(n+1)\int_{i/(n+1)}^{(i+1)/(n+1)}\left[g(t) - g\left(\frac{i}{n}\right)\right]dt \\ &\qquad + b_i(n+1)\int_{(i+1)/(n+1)}^{(i+2)/(n+1)}\left[g(t) - g\left(\frac{i}{n}\right)\right]dt. \end{aligned}$$

We have

$$
\begin{aligned}
&\|T_n g - B_n g\|_p^p \\
&\quad \le \sum_{i=0}^{n} \int_0^1 P_{n,i}(x)\,dx\, 3^p(n+1)\Bigg\{|c_i|^p \int_{(i-1)/(n+1)}^{i/(n+1)} \left|\int_{i/n}^{t} g'(u)\,du\right|^p dt \\
&\qquad + |a_i|^p \int_{i/(n+1)}^{(i+1)/(n+1)} \left|\int_{i/n}^{t} g'(u)\,du\right|^p dt \\
&\qquad + |b_i|^p \int_{(i+1)/(n+1)}^{(i+2)/(n+1)} \left|\int_{i/n}^{t} g'(u)\,du\right|^p dt\Bigg\} \\
&\quad \le 3^p \sum_{i=0}^{n} \Bigg\{|c_i|^p \int_{(i-1)/(n+1)}^{i/(n+1)} \int_{(i-1)/(n+1)}^{(i+1)/(n+1)} |g'(u)|^p\,du \left(\frac{2}{n+1}\right)^{p-1} dt \\
&\qquad + |a_i|^p \int_{i/(n+1)}^{(i+1)/(n+1)} \int_{i/(n+1)}^{(i+1)/(n+1)} |g'(u)|^p\,du \left(\frac{1}{n+1}\right)^{p-1} dt \\
&\qquad + |b_i|^p \int_{(i+1)/(n+1)}^{(i+2)/(n+1)} \int_{i/(n+1)}^{(i+2)/(n+1)} |g'(u)|^p\,du \left(\frac{2}{n+1}\right)^{p-1} dt\Bigg\} \\
&\quad \le \left(\frac{6}{n+1}\right)^p \sum_{i=0}^{n} \Bigg\{|c_i|^p \int_{(i-1)/(n+1)}^{(i+1)/(n+1)} |g'(u)|^p\,du \\
&\qquad + |a_i|^p \int_{i/(n+1)}^{(i+1)/(n+1)} |g'(u)|^p\,du + |b_i|^p \int_{i/(n+1)}^{(i+2)/(n+1)} |g'(u)|^p\,du\Bigg\} \\
&\quad \le \left(\frac{6}{n+1}\right)^p \{2\|g'\|_p^p + 2\|g'\|_p^p + 2\|g'\|_p^p\} \\
&\quad \le \left(\frac{36}{n+1}\|g'\|_p\right)^p.
\end{aligned}
$$

Thus, we have

$$\|T_n g - B_n g\|_p \le \frac{36}{n+1}\|g'\|_p.$$

The inequality

$$\|g'\|_p \le C_p\big(\|g\|_p + \|\varphi^2 g''\|_p\big), \qquad 1 \le p < \infty \tag{6.5}$$

which is due to V. Totik [7], yields our estimate (6.4).

The proof of our Theorem 2 in the case $1 \le p < \infty$ is complete.

In the case $p = \infty$, things are different. Inequality (6.5) is not valid any longer. Hence (6.3) does not hold in general. So we use a K-functional introduced by one of the authors in [10] to prove this case.

Our K-functional is defined in $C[0,1]$ as

$$K_{1,2}(f,t)_\infty = \inf_{g' \in A.C._{loc}} \{\|f-g\|_\infty + t(\|g\|_\infty + \|g'\|_\infty + \|\varphi^2 g''\|_\infty)\}. \quad (6.6)$$

By this K-functional we will complete our proof.

Proof of Theorem 2 for $p = \infty$. We first prove for $f \in C[0,1]$, $0 < \alpha < 1$, that $\|T_n f - f\|_\infty = O(n^{-\alpha})$ if and only if $K_{1,2}(f,t)_\infty = O(t^\alpha)$. By a result of A. Grundmann [5] (see also [4]) it is sufficient to show that

$$\|T_n f\|_\infty + \|T_n'(f)\|_\infty + \|\varphi^2 T_n''(f)\|_\infty \le Mn\|f\|_\infty, \qquad \text{for } f \in C[0,1]; \quad (6.7)$$

$$\|T_n f\|_\infty + \|T_n'(f)\|_\infty + \|\varphi^2 T_n''(f)\|_\infty \le M(\|f'\|_\infty + \|\varphi^2 f''\|_\infty + \|f\|_\infty),$$
$$\text{for } f \in C[0,1],\ f' \in A.C._{loc}; \quad (6.8)$$

$$\|T_n g - g\|_\infty \le \frac{C}{n}(\|g\|_\infty + \|g'\|_\infty + \|\varphi^2 g''\|_\infty),$$
$$\text{for } g \in C[0,1],\ g' \in A.C._{loc}. \quad (6.9)$$

Relations (6.7) and (6.8) are stated in Lemmas 4.1 and 4.2 except the estimate of $\|T_n'(f)\|_\infty$ which is simpler. We omit it.

Note that

$$\|B_n g - g\|_\infty \le \frac{1}{n}\|\varphi^2 g''\|_\infty, \qquad \text{for } g \in C[0,1],\ g' \in A.C._{loc}.$$

The proof of (6.9) is also easy since we have

$$\begin{aligned}
&\|T_n g - B_n g\|_\infty \\
&\quad \le \sup_x \left\{ \sum_{i=0}^{n} P_{n,i}(x) \left\{ |c_i| \int_{(i-1)/(n+1)}^{i/(n+1)} |t - \frac{i}{n}|\, dt \right.\right. \\
&\qquad\qquad + |a_i| \int_{i/(n+1)}^{(i+1)/(n+1)} \left| t - \frac{i}{n} \right| dt \\
&\qquad\qquad \left.\left. + |b_i| \int_{(i+1)/(n+1)}^{(i+2)/(n+1)} \left| t - \frac{i}{n} \right| dt \right\} (n+1)\|g'\|_\infty \right\} \\
&\quad \le \frac{9}{n+1}\|g'\|_\infty.
\end{aligned}$$

Thus, we have proved the equivalence of the rate of convergence and the behavior of the K-functional. It is easy to check that (6.7), (6.8), and (6.9) are also valid for the Bernstein operators (1.1) which implies that $K_{1,2}(f,t)_\infty = O(t^\alpha)$ if and only if $\|B_n f - f\|_\infty = O(n^{-\alpha})$. Then the characterization theorem of the Bernstein operators [3] implies that (3.3) and (3.4) are equivalent for $p = \infty$. Hence our Theorem 2 holds. The proof of Theorem 2 is complete.

Remark. Concerning the different saturation conditions of the Bernstein polynomials and the Kantorovich operators it must be interesting to solve the saturation problem for our class of operators $\{T_n(f,x)\}_{n\in N}$. We also expect that this will depend on the choice of the matrices.

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